Real analysis

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1 Examples in class

Because the scope of the exam is small, there are not many examples, and the convergence section is not included.

1. Cantor/ Cantor-like sets and properties, Cantor-Lebesgue function;
2. Monotonically decreasing set's measure continuous? $E_i = \{(x, y) | |y| \leq \frac{1}{i}\}$;
3. Borel sets, Lebesgue measurable sets, $2^{\mathbb{R}^d}$, Vitali set;
4. Egorov for $m(E) = \infty$? $f_k = \chi_{[-k,k]}$;

2 Examples and properties from the exercises

From exercises in Stein's book: The exercises in this book are either examples or counter-examples of properties in class, or you can learn something new by doing them.

1. Cantor (Chapter1 Ex1-4).
2. An example of an open set O with the following property: the boundary of the closure of O has positive Lebesgue measure.(Chapter1 Ex9)
3. A decreasing sequence of positive continuous functions on the interval [0, 1], whose pointwise limit is not Riemann integrable. (Chapter1 EX10)
4. Every open set in $\mathbb{R}$ is the disjoint union of open intervals, but in $\mathbb{R}^d$ it is false(Chapter1 Ex12).
5. Learn about F_σ and G_δ sets. Open? Close? Borel?(Chapter1 Ex13).
6. Covering by a finite number of intervals will not suffice in the definition of the outer measure.
7. Define the outer measure by taking coverings by rectangles(Chapter1 Ex15).
8. Regarding the set operation $A + B$ (Chapter1 Ex19,20).

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9. There is a continuous function that maps a Lebesgue measurable set to a non-measurable set(Chapter1 Ex21).

10. Cantor, Non-measurable, something holds for measurable sets do not hold;(Chapter1 Ex31-35).

11. Zero-measure curve (Chapter1 Ex37, Chapter2 Ex7).

12. Continuity of $d(x, F)$(Chapter2 Ex5);

13. Integrability of f on R does not necessarily imply the convergence of $f(x)$ to 0 as $x \to \infty$;(Chapter2 Ex6)

14. Tchebychev inequality;(Chapter2 Ex9)

15. Give an example of two measurable sets A and B such that A + B is not measurable.(Chapter2 Ex13)

16. Integrable but unbounded in any interval;(Chapter2 Ex15)

17. Riemann-Lebesgue lemma (Chapter2 Ex22);

3 Knowlege frame

3.1 Measure

- Outer measure
  - Volume for rectangles
  - Properties
    * Countable subadditivity
    * External regularity
    * Positive distance additivity
    * Cut cubes without losing volume
    * Translation invariance
- Measure
  - Equivalent definition
  - Measurable set structure: F_σ, G_δ
  - Properties
    * B-C lemma
    * Countable additivity
    * Continuity of measure
    * Compact set approximation
    * Finite square symmetric difference approximation
    * Invariance

3.2 Measurable functions

- Equivalent condition
- Approximation theorem: simple, step,pointwise, a.e.,monotonous, ...
- Convergence analysis
- Theorem: Egorov, Lusin

3.3 Lebesgue Integral

- Construction: simple $\to$ nonnegative $\to$ integrable
- Commutative order theorem: MCT, Fatou, DCT, Tonelli, Fubini
- Properties
  - Absolute continuity
  - Translational continuity
  - Small in the distance and high places, i.e. $\int_E |f| \leq \varepsilon$ for $E = \{f \leq M\}$ or $E = B^c$ where B is a large ball.